AIAA JOURNAL
Vol. 43, No. 11, November 2005

Nonlinear Perturbation Theory for Structural Dynamic Systems

Hua-Peng Chen*
University of Glasgow, Glasgow, Scotland G12 8LT, United Kingdom

A novel nonlinear perturbation theory for structural dynamic systems is developed that can provide an exact relationship between the perturbation of structural parameters and the perturbation of modal parameters. A system of governing equations based on the developed theory is further derived, which can be utilized for general applications, such as structural reanalyses, eigendata modification, model updating, and damage identification, suitable for all types of structures including mechanical systems, framed structures, and continua. Neither model reduction nor mode shape expansion is required for modal updating and damage identification because information about incomplete measured modal data can be directly employed. The developed theory successfully avoids adopting a Taylor series expansion procedure and then the derivatives of modal parameters are not required. Computational procedures based on the derived nonlinear governing equations are presented for eigendata modification, model updating, and damage identification. The Jacobi transformation method and the accelerated modal method are introduced to make the proposed techniques particularly suitable for cases with a very large perturbation of structural parameters. Finally, two numerical examples are given to demonstrate the effectiveness of the proposed techniques. The results show that the modified modal parameters can be predicted exactly even for cases with a large modification of structural parameters, and the analytical model can be adjusted correctly using the information about limited modal data available.

Nomenclature

$a^{K^*}_{iji}$, $a^{M^*}_{iji}$, a^{K}_{kjl}, a^{M}_{kjl}	=	sensitivity coefficients for model updating
$\boldsymbol{a}^{K^*}_{ji}$, $\boldsymbol{a}^{M^*}_{ji}$	=	sensitivity coefficient vectors for model updating
b^K_i, b^M_i	=	sensitivity coefficients for eigendata modification
C_{ik}	=	mode participation factor for the kth element of the ith mode
$\boldsymbol{D}$, $d^{(i)}_{kk}$, $d^{(i)}_{kl}$	=	sensitivity coefficient matrix and sensitivity coefficients for evaluating mode participation factors
$\boldsymbol{J}$	=	Jacobi rotation matrix for all transformations
$\boldsymbol{K}$, $\boldsymbol{K}^*$	=	global stiffness matrices, original and modified
$\boldsymbol{K}_j$	=	contribution of the jth element to global stiffness matrix
$\boldsymbol{M}$, $\boldsymbol{M}^*$	=	global mass matrices, original and modified
$\boldsymbol{M}_j$	=	contribution of the jth element to global mass matrix
N	=	total number of degrees of freedom (DOF)
NA	=	total number of the measured DOF readings of total NL modified modes available
NAI	=	total number of DOF readings available
NC	=	total number of original eigenvectors considered
NL	=	total number of modified modes available
NP	=	total number of system parameters
P_m, $p^{(m)}_{st}$	=	Jacobi rotation matrix and element for the mth transformation
$\boldsymbol{r}^*_i$	=	vector of residual forces for the ith modified mode
y	=	norm of the vector of residual forces $\boldsymbol{r}^*_i$
α_j, β_j	=	system parameters for the jth element
Δ	=	difference of quantities between modified and original systems
ϕ_i, ϕ^*_i	=	ith mode shapes or eigenvectors, original and modified
ϕ^a_i, ϕ^u_i	=	ith original eigenvectors restricted to the dimension for the DOF readings available and for the DOF readings unknown
ϕ^{a*}_i, ϕ^{u*}_i	=	ith modified eigenvectors restricted to the dimension for the DOF readings available and for the DOF readings unknown
φ^a_i	=	vector combining DOF readings available and corresponding original eigenvector for the remaining dimension
λ_i, λ^*_i	=	ith eigenvalues of characteristic equation, original and modified
μ_i	=	acceleration factor for the ith mode
θ	=	Jacobi rotation angle
υ_i	=	mode scale factor for the ith measured mode
ω_i, ω^*_i	=	natural frequencies of the ith mode, original and modified
ψ^{a*}_i	=	DOF readings available for the ith modified mode

Superscripts

$'$	=	transformed quantity
-	=	quantity associated with accelerated factor
$*$	=	quantity for the modified system
T	=	transpose of a matrix quantity

I. Introduction

THE relationship between structural parameters, for example, stiffness and mass, and modal parameters, for example, natural frequencies and mode shapes, for a structural dynamic system can be represented using the characteristic equation of the system. The analytical modal parameters then can be calculated from the characteristic equation for the numerical representation, which is usually based on a finite element model. From the characteristic equation, it is obvious that changes in the structural parameters will cause changes in modal parameters. To predict well the changes caused, the exact relationship between the perturbation of structural parameters and the perturbation of modal parameters should be established, which can be applied to various practical engineering problems, such as structural reanalyses, eigendata modification, model updating, damage identification, etc.

The structural reanalyses and eigendata modification techniques are often employed to obtain the optimum condition of a numerical model of a structural dynamic system by altering the dynamic response. In a number of studies, such as that of Fox and Kapoor[1]

Received 21 December 2004; revision received 20 June 2005; accepted for publication 21 June 2005.

*Researcher, Department of Civil Engineering.

and Rudisill and Bhatia,[2] sensitivity analyses based on eigenvalue and eigenvector derivatives were utilized for the dynamic reanalyses of a structural dynamic system. However, the efficiency of these sensitivity analysis methods is limited because these methods are complicated and suitable for small modifications of structural parameters.[3] To overcome this usual shortcoming, To and Ewins[4] present a procedure for determining the revised modal parameters through a nonlinear sensitivity analysis. The modified eigenvalues are determined using the stationary property of the Rayleigh quotient. It was mentioned in their paper that the proposed approach may not converge when the first approximation for a modified structure's eigenvalue is closer to a neighboring eigenvalue than to the modified one. Beliveau et al.[5] developed an iterative solution scheme for calculating the eigenvector sensitivity, where a least-squares formulation for the eigenvector sensitivity is utilized. It was shown that the proposed iterative procedure converged for the sensitivity of eigenvectors.

Structural model updating methods are often utilized to adjust the analytical model using the measured modal data to maximize the correlation between the analytical and testing model. Model updating methods based on sensitivity analysis are usually employed for updating a numerical model, such that it can exactly reproduce an incomplete set of measured eigendata. In the works of Berman and Nagy[6] and Caesar and Peter,[7] structural parameters (stiffness and/or mass) are updated separately by minimizing an objective function, with constraints imposed through Lagrange multipliers. Kabe[8] proposed a minimization of the objective function subject to symmetric Lagrange multiplier constraints, where the structural connectivity is preserved. However, to find the Largrange multipliers is computationally expensive. Kammer[9] presented a projector matrix method that is computationally efficient and turns out to be equivalent to Kabe's method in most cases. Furthermore, Smith and Beattie[10] considered quasi-Newton methods for stiffness updating that preserve the structural connectivity. Ladeveze et al.[11] developed a method based on the computation of the error measure on the constitutive relation to correct both the stiffness and mass matrices. Recently, techniques used for model updating have been widely adopted for structural damage identification because the techniques for model updating are often closely related to those for damage identification using measured vibration modal data.

Note that the first-order approximation used for model updating may perform properly when the changes in structural parameters from the initial model to the refined model are small. However, for model refinement with relatively large modifications of structural parameters, the first-order approximation may be inaccurate because a large parameter change needs to be adjusted. Here, a general nonlinear perturbation theory is developed that can provide an exact relationship between the perturbations of structural parameters and the associated modal parameters. Based on the developed general theory, various governing equations are derived, which can be utilized for general applications, such as eigendata modification and model updating, as well as damage identification. The results of numerical examples show that the iterative procedures discussed here converge quickly for both eigendata modification and model updating problems. The proposed techniques for eigendata modification can be employed to determine the exact modal properties of the modified system even in the cases where large modifications of structural parameters exist. Meanwhile, the proposed techniques for model updating and damage identification can be utilized to adjust the analytical model or identify damage in a structure correctly using information about only a limited number of modified natural frequencies or using information about incomplete model data with a limited number of degrees of freedom (DOF) readings.

II. Nonlinear Perturbation Theory

The characteristic equations for the original and the modified structural dynamic system can be expressed as

$$(K - \lambda_i M)\phi_i = 0 \tag{1}$$

$$\left(K^* - \lambda_i^* M^*\right)\phi_i^* = 0 \tag{2}$$

Suppose that the modifications of stiffness matrix and mass matrix are defined as $\Delta\boldsymbol{K}$ and $\Delta\boldsymbol{M}$, respectively. The stiffness matrix and mass matrix for the modified structural system, therefore, can be written as

$$K^* = K + \Delta K \tag{3}$$

$$M^* = M + \Delta M \tag{4}$$

Meanwhile, the perturbations of the ith eigenvalue and the corresponding eigenvector, which are caused by the perturbations of stiffness matrix and mass matrix, are defined as $\Delta\lambda_i$ and $\Delta\phi_i$, respectively. Note that the eigenvectors for both the original and the modified structural systems are linearly independent because the stiffness and mass matrices are symmetric. The ith eigenvalue and the corresponding eigenvector for the modified structural system, therefore, are

$$\lambda_i^* = \lambda_i + \Delta\lambda_i \tag{5}$$

$$\phi_i^* = \phi_i + \Delta\phi_i \tag{6}$$

Premultiplying Eq. (1) by ϕ_i^{*T}, and using the transpose of the equation, yields

$$\phi_i^T K\phi_i^* = \lambda_i\phi_i^T M\phi_i^* \tag{7}$$

Similarly, premultiply Eq. (1) by ϕ_k^{*T}, where $k \neq i$, then rewrite the equation as

$$\phi_k^T K\phi_i^* = \lambda_k\phi_k^T M\phi_i^* \tag{8}$$

Premultiplying Eq. (2) by ϕ_i^T, and using Eqs. (3–7), leads to

$$\phi_i^T[\Delta K - (\lambda_i + \Delta\lambda_i)\Delta M](\phi_i + \Delta\phi_i) - \Delta\lambda_i\phi_i^T M(\phi_i + \Delta\phi_i) = 0 \tag{9}$$

Furthermore, premultiplying Eq. (2) by ϕ_k^T, when k is not equal to i, and using Eqs. (3–6) and (8) yields

$$\phi_k^T[\Delta K - (\lambda_i + \Delta\lambda_i)\Delta M](\phi_i + \Delta\phi_i) - (\lambda_i + \Delta\lambda_i - \lambda_k)\phi_k^T M(\phi_i + \Delta\phi_i) = 0 \tag{10}$$

From Eqs. (9) and (10), it can be seen that if $[\Delta K - (\lambda_i + \Delta\lambda_i)\Delta M]$ does not equal to zero, then $\Delta\lambda_i$ and (or) $\Delta\phi_i$ will not be zero, whereas if $\Delta\lambda_i$ and (or) $\Delta\phi_i$ do not equal to zero, where i ranges from 1 to N, then it can also be obtained that ΔK and (or) ΔM will not be zero.

It is assumed that the mode shapes of the original structural system are mass normalized in the following form, where $k \neq i$:

$$\phi_i^T M\phi_i = 1 \tag{11}$$

$$\phi_k^T M\phi_i = 0 \tag{12}$$

Meanwhile, to ensure the uniqueness of a mode shape of the modified structural system, we can assume that the modified eigenvector is mass normalized in the form defined as follows because any scalar multiple of a modified eigenvector is also a modified eigenvector:

$$\phi_i^T M\phi_i^* = 1 \tag{13}$$

It can be shown later that the preceding assumptions of mode shape normalizations for the original system and for the modified system make the proposed theory simple and possible.

Equation (13) can be rewritten as follows by using Eqs. (6) and (11):

$$\phi_i^T M\Delta\phi_i = 0 \tag{14}$$

Define mode participation factors C_{ik}, where $k \neq i$, as

$$C_{ik} = \phi_k^T M\phi_i^* \tag{15}$$

Equations (9) and (10), by using Eqs. (13) and (15), are now rewritten as, respectively,

$$\phi_i^T[\Delta K - (\lambda_i + \Delta\lambda_i)\Delta M]\{\phi_i + \Delta\phi_i\} - \Delta\lambda_i = 0 \qquad (16)$$

$$\phi_k^T[\Delta K - (\lambda_i + \Delta\lambda_i)\Delta M]\{\phi_i + \Delta\phi_i\} - (\lambda_i + \Delta\lambda_i - \lambda_k)C_{ik} = 0 \qquad (17)$$

From the preceding paragraphs, Eq. (16) could be considered as a special case of the general equation (17), with the definition of $C_{ii} = 1$ when $k = i$.

The mode participation factors, where $k \neq i$, from Eqs. (15) and (12), are expressed as

$$C_{ik} = \phi_k^T M \Delta\phi_i \qquad (18)$$

Premultiplying Eq. (18) by ϕ_k, then summing up the equations from 1 to N and using the mass normalization of the original eigenvectors, Eqs. (11) and (12), and the assumption of linearly independent eigenvectors of the original system, finally we have

$$\Delta\phi_i = \sum_{k=1,k\neq i}^{N} C_{ik}\phi_k \qquad (19)$$

From Eq. (17), the mode participation factors C_{ik} can be calculated from

$$C_{ik} = \frac{\phi_k^T[\Delta K - (\lambda_i + \Delta\lambda_i)\Delta M]\{\phi_i + \Delta\phi_i\}}{(\lambda_i + \Delta\lambda_i - \lambda_k)} \qquad (20)$$

Consequently, the modification of eigenvectors for a structural system can be obtained from

$$\Delta\phi_i = \sum_{k=1,k\neq i}^{N} \frac{\phi_k^T[\Delta K - (\lambda_i + \Delta\lambda_i)\Delta M]\{\phi_i + \Delta\phi_i\}}{(\lambda_i + \Delta\lambda_i) - \lambda_k}\phi_k \qquad (21)$$

It is found from Eqs. (19) and (21) that the modification of an eigenvector of a structural system can be expressed as the linear combination of the original eigenvectors except the corresponding original one. When k is large enough, the terms with subscripts greater than k can be neglected. Therefore, N can be suitably replaced by NC, and Eq. (19) is rewritten as

$$\Delta\phi_i = \sum_{k=1,k\neq i}^{\mathrm{NC}} C_{ik}\phi_k \qquad (22)$$

In Eq. (22), it is assumed that the modification of an eigenvector could be constructed from a linear combination of a subset of original mode shapes because NC is a subset of the total number of all modes available, N. The difficulty in computing all eigenvectors of the original system could then be avoided in the cases where a significant number of DOF are adopted for modeling the dynamic system.

The given general nonlinear perturbation theory, which represents the exact relationship between the perturbation of structural parameters and the perturbation of modal parameters, can be further developed for various applications, such as structural reanalyses, eigendata modification, model updating, and damage identification.[12,13] Note that Taylor series expansion procedure is not employed and information about the derivatives of eigenvalues or eigenvectors is not required to develop the theory.

When the modifications of structural parameters are small enough, only the first-order approximation may be sufficient. The set of nonlinear Eqs. (16) and (21), then, can be simplified to linear relationship in the form

$$\Delta\lambda_i = \phi_i^T[\Delta K - \lambda_i \Delta M]\phi_i \qquad (23)$$

$$\Delta\phi_i = \sum_{k=1,k\neq i}^{N} \frac{\phi_k^T[\Delta K - \lambda_i\Delta M]\phi_i}{(\lambda_i - \lambda_k)}\phi_k \qquad (24)$$

The preceding linear relationship is very commonly utilized for sensitivity analysis, model updating, and damage identification, such as in the works of Link[14] and Cawley and Adams.[15] Note that the set of linear equations might be insufficient if relatively large modifications of structural parameters are present.

III. Eigendata Modification

When the modifications of structural parameters $\mathbf{\Delta K}$ and $\mathbf{\Delta M}$ are known, the modifications of modal parameters (eigendata), $\Delta\lambda_i$ and $\Delta\phi_i$, can be computed using the developed nonlinear perturbation theory, which is considered as a forward problem here. Note that information on the modal parameters of the modified system is not required during the evaluation of the modification of eigendata.

The modification of eigenvalues, after rewriting Eq. (16), can be computed from

$$\Delta\lambda_i = \frac{\phi_i^T[\Delta K - \lambda_i\Delta M]\{\phi_i + \Delta\phi_i\}}{1 + \phi_i^T \Delta M\{\phi_i + \Delta\phi_i\}} \qquad (25)$$

and by using Eqs. (20) and (22), the mode participation factors C_{ik}, which are used for calculating the modification of eigenvectors, can be obtained from

$$C_{ik} = \left\{\phi_k^T[\Delta K - (\lambda_i + \Delta\lambda_i)\Delta M]\phi_i + \sum_{l=1,l\neq i,k}^{\mathrm{NC}} \phi_k^T[\Delta K - (\lambda_i + \Delta\lambda_i)\Delta M]\phi_l C_{il}\right\} \Big/ \left\{(\lambda_i + \Delta\lambda_i - \lambda_k) - \phi_k^T[\Delta K - (\lambda_i + \Delta\lambda_i)\Delta M]\phi_k\right\} \qquad (26)$$

Rewriting Eqs. (25) and (26) yields

$$\Delta\lambda_i = \frac{b_i^K - \lambda_i b_i^M}{1 + b_i^M} \qquad (27)$$

$$C_{ik} = \frac{d_{ki}^{(i)}}{d_{kk}^{(i)}} + \sum_{l=1,l\neq i,k}^{\mathrm{NC}} \frac{d_{kl}^{(i)}}{d_{kk}^{(i)}} C_{il} \qquad (28)$$

where sensitivity coefficients b_i^K, b_i^M, $d_{kl}^{(i)}$, and $d_{kk}^{(i)}$ are defined in a general form as

$$b_i^K = \phi_i^T \Delta K \phi_i^*, \qquad b_i^M = \phi_i^T \Delta M \phi_i^* \qquad (29)$$

$$d_{kl}^{(i)} = d_{lk}^{(i)} = \phi_k^T\left(\Delta K - \lambda_i^* \Delta M\right)\phi_l$$

$$d_{kk}^{(i)} = \lambda_i^* - \lambda_k - \phi_k^T\left(\Delta K - \lambda_i^*\Delta M\right)\phi_k \qquad (30)$$

An iterative procedure for calculating the modification of eigenvalues $\Delta\lambda_i$ and the mode participation factors C_{ik} is introduced because the two governing equations (25) and (28) are coupled. Note that the iterative procedure for calculating C_{ik} is very similar to the Gauss–Seidel scheme for numerical solution of linear equations. The Gauss–Seidel scheme may or may not converge if the coefficient matrix is not diagonally dominant. Therefore, the magnitude of each leading diagonal element should exceed the sum of the magnitudes of the remaining elements in the same row of the matrix of coefficients defined in Eq. (30), to improve convergence performance for the iterative procedure given in Eq. (28).

A. Jacobi Transformation Method

The Jacobi transformation method has been developed for the solution of eigenproblems for symmetric matrices. The basic scheme of the method is to reduce a symmetric matrix into diagonal form using successive pre- and postmultiplication by a rotation matrix, which is selected in such way that an off-diagonal element in the symmetric matrix is zeroed, leading to the symmetric matrix closer to diagonal form. Therefore, the basic idea of the Jacobi transformation method can be adopted so that the symmetric coefficient matrix

$\boldsymbol{D}$ defined in Eq. (30) for evaluating mode participation factors becomes diagonal dominant.

When it is assumed that the diagonal element in kth row does not dominate the remaining elements, the largest off-diagonal element in kth row then is selected, for example, element d_{kl}. To zero the selected element d_{kl}, a rotation matrix $\boldsymbol{P}_m$ for the mth transformation is defined as

$$P_m = \left\{ p_{st}^{(m)} \right\} \tag{31}$$

where $p_{st}^{(m)} = 1$, if $s = t$, except $p_{kk}^{(m)} = p_{ll}^{(m)} = \cos\theta$; $p_{st}^{(m)} = 0$, if $s \neq t$, except $p_{kl}^{(m)} = -p_{lk}^{(m)} = -\sin\theta$; $\boldsymbol{P}_m$ is an orthogonal matrix, and the rotation angle θ is selected from the condition that the element d_{kl} in kth row be zero and calculated from

$$\tan 2\theta = \frac{2d_{kl}^{(m)}}{d_{kk}^{(m)} - d_{ll}^{(m)}} \tag{32}$$

and the evaluation of the symmetric coefficient matrix after the mth transformation, $\boldsymbol{D}_{m+1}$, then can be computed from

$$D_{m+1} = P_m^T D_m P_m \tag{33}$$

Note that the numerical evaluation of the coefficient matrix requires only the linear combination of the corresponding two rows and two columns.

The rows in the coefficient matrix are tested sequentially, and a rotation is only applied if the leading diagonal element of the tested row does not dominate the remaining elements. When r is assumed to be the last rotation, then

$$J = P_1 P_2 \cdots P_r \tag{34}$$

and the coefficient matrix after the last evaluation becomes diagonal dominant, that is,

$$D' = J^T D J \tag{35}$$

Consequently, the sensitivity coefficients $d_{kl}^{(i)}$ and $d_{kk}^{(i)}$, defined in Eq. (30), are replaced by the evaluated values $d_{kl}^{\prime(i)}$ and $d_{kk}^{\prime(i)}$, calculated from Eq. (35), and then Eq. (28) is rewritten here as

$$C'_{ik} = \frac{d_{ki}^{\prime(i)}}{d_{kk}^{\prime(i)}} + \sum_{l=1, l\neq i,k}^{\mathrm{NC}} \frac{d_{kl}^{\prime(i)}}{d_{kk}^{\prime(i)}} C'_{il} \tag{36}$$

and the mode participation factors C_{ik} can obtained from

$$C_{ik} = J C'_{ik} \tag{37}$$

B. Accelerated Modal Method

To improve the convergence performance of the solution of the set of nonlinear equations and to avoid the problems associated with the interchange of the predicted eigenvalues, an acceleration factor μ is selected by minimizing the residual force of the modified structural system.

Once the change of eigenvalue $\Delta\lambda_i$ and the increment of mode participation factors ΔC_{ik} are obtained using the described techniques, the new evaluations for the modified eigenvalue $\bar{\lambda}_i^*$ and the mode participation factors $\bar{C}_{ik}$, considering the acceleration factor μ_i, can be defined as

$$\bar{\lambda}_i^* = \lambda_i + \mu_i \Delta\lambda_i \tag{38}$$

$$\bar{C}_{ik} = C_{ik} + \mu_i \Delta C_{ik} \tag{39}$$

and the corresponding modified eigenvector $\bar{\phi}_i^*$, after using Eqs. (6) and (22), is

$$\bar{\phi}_i^* = \phi_i + \sum_{k=1, k\neq i}^{\mathrm{NC}} \bar{C}_{ik} \phi_k \tag{40}$$

Considering the characteristic equation for the modified structural dynamic system, that is, Eq. (2), and using Eqs. (38–40), a residual force for the ith modified mode is defined as

$$r_i^* = r_i^*(\mu_i) = \left(K^* - \bar{\lambda}_i^* M^*\right) \bar{\phi}_i^* \tag{41}$$

The norm y of the residual force is then defined as

$$y = y(\mu_i) = r_i^{*T}(\mu_i) r_i^*(\mu_i) \tag{42}$$

The acceleration factor μ_i is then calculated by minimizing the norm defined earlier.

The pairing of the eigenmodes for the original and the modified structural dynamic systems can be checked using the modal assurance criterion (MAC) factors defined as

$$\mathrm{MAC}(k, i) = \frac{\left|\phi_k^T \phi_i^*\right|^2}{\left|\phi_k^T \phi_k\right| \left|\phi_i^{*T} \phi_i^*\right|} \tag{43}$$

The highest MAC(k, i) factors indicate the most possible pairings of the original mode k and the modified mode i.

The preceding formulation will be applied to develop an iterative solution procedure used for evaluating eigendata modification.

The procedure is initiated by assuming that the initial mode participation factors C_{ik} are zero. Physically, this implies that the initial modified eigenvalues are obtained from the assumption that the modified eigenvectors are identical to the original ones. A first approximation for the modified eigenvalues λ_i^* is then obtained from Eqs. (5) and (25). After the initial modified eigenvalues λ_i^* are obtained, the next approximation for the mode participation factors C_{ik} is calculated using Eq. (28). To improve the convergence performance of the solution, the Jacobi transformation method might be adopted to calculate the mode participation factors C_{ik} using Eq. (36), and the accelerated modal method could be employed to evaluate a new approximation for λ_i^* and C_{ik} by minimizing the norm defined in Eq. (42). Therefore, the process is used recursively to obtain further approximation for λ_i^* and C_{ik}, and the recursive process is repeated until convergence for the modified eigenvalues λ_i^* is achieved.

IV. Model Updating and Damage Identification

When the perturbations of modal parameters, $\Delta\lambda_i$ and/or $\Delta\phi_i$, are known, the perturbations of structural parameters $\boldsymbol{\Delta K}$ and $\boldsymbol{\Delta M}$ can also be inversely determined using the developed nonlinear perturbation theory, which is considered as an inverse problem here. Different procedures are proposed for model updating and damage identification depending on information about modal data available.

System parameters, such as coefficients of stiffness or mass matrix and parameters for material properties and geometric properties, are employed to represent the modifications of structural parameters, for example, stiffness matrix and/or mass matrix. These system parameters characterizing either a matrix coefficient level, a Gauss point level, an element level, or a subsystem level can be utilized for model updating and damage identification.

Here, it is assumed that system parameters characterize an element level; the perturbations of structural stiffness matrix and mass matrix are then defined as

$$\Delta K = \sum_{j=1}^{\mathrm{NP}} \alpha_j K_j \tag{44}$$

$$\Delta M = \sum_{j=1}^{\mathrm{NP}} \beta_j M_j \tag{45}$$

A. Information on Only $\boldsymbol{\lambda_i^*}$ Available

A set of nonlinear equations have to be utilized to solve for the modifications of structural parameters because only a total number of NL modified eigenvalues are known and the eigenvectors for the modified structural system are not available.

Rewriting Eq. (16) leads to

$$\phi_i^T \left[\Delta K - \lambda_i^* \Delta M\right] \phi_i^* + \Delta\lambda_i = 0 \tag{46}$$

where the modified eigenvectors ϕ_i^* are calculated using Eqs. (6) and (22), that is,

$$\phi_i^* = \phi_i + \sum_{k=1,k\neq i}^{\mathrm{NC}} C_{ik}\phi_k \tag{47}$$

where the mode participation factors C_{ik} are obtained from Eq. (26), rewritten here as

$$C_{ik} = \frac{\phi_k^T\left(\Delta K - \lambda_i^*\Delta M\right)\phi_i + \sum_{l=1,l\neq i,k}^{\mathrm{NC}} \phi_k^T\left(\Delta K - \lambda_i^*\Delta M\right)\phi_l C_{il}}{\lambda_i^* - \lambda_k - \phi_k^T\left(\Delta K - \lambda_i^*\Delta M\right)\phi_k} \tag{48}$$

When Eqs. (44) and (45) are used, the set of nonlinear Eqs. (46) and (48) can be rewritten as

$$\sum_{j=1}^{\mathrm{NP}} a_{iji}^{K^*}\alpha_j - \lambda_i^* \sum_{j=1}^{\mathrm{NP}} a_{iji}^{M^*}\beta_j - \Delta\lambda_i = 0 \tag{49}$$

$$C_{ik} = \left[\sum_{j=1}^{\mathrm{NP}}\left(a_{kji}^K\alpha_j - \lambda_i^* a_{kji}^M\beta_j\right) + \sum_{j=1}^{\mathrm{NP}}\sum_{l=1,l\neq i,k}^{\mathrm{NC}}\left(a_{kjl}^K\alpha_j - \lambda_i^* a_{kjl}^M\beta_j\right)C_{il}\right] \Bigg/ \left[\lambda_i^* - \lambda_k - \sum_{j=1}^{\mathrm{NP}}\left(a_{kjk}^K\alpha_j - \lambda_i^* a_{kjk}^M\beta_j\right)\right] \tag{50}$$

where the sensitivity coefficients associated with eigenmode and structural parameters are defined in a general form as

$$a_{iji}^{K^*} = \phi_i^T K_j \phi_i^*, \qquad a_{iji}^{M^*} = \phi_i^T M_j \phi_i^* \tag{51a}$$

$$a_{kjl}^{K} = \phi_k^T K_j \phi_l, \qquad a_{kjl}^{M} = \phi_k^T M_j \phi_l \tag{51b}$$

A computational procedure similar to that used for eigendata modification problems is developed here to solve for the system parameters α_j and β_j and the mode participation factors C_{ik} using the preceding formulation. Again, the initial mode participation factors C_{ik} are assumed to be zero. A first approximation for the system parameters α_j and β_j is then obtained from Eq. (49). Depending on the total number of the modified natural frequencies available NL (number of equations) and the total number of system parameters present $2 * \mathrm{NP}$ (number of unknowns), the sensitivity coefficient matrix in Eq. (49) may not be square. To find a solution for what is in general an ill-conditioned system, the filtered singular value decomposition (SVD) technique[16] is employed to estimate the system parameters α_j and β_j.

After the initial system parameters α_j and β_j are obtained, the next approximation for the mode participation factors C_{ik} can be calculated from Eq. (50). The Jacobi transformation method discussed earlier might be needed to evaluate the mode participation factors C_{ik}. The modified eigenvectors ϕ_i^* then are calculated using Eq. (47). Consequently, Eqs. (49) and (50), as well as Eq. (47), are used recursively to compute further approximation for α_j and β_j as well as C_{ik}, and the preceding recursive process is repeated until the convergence for system parameters α_j and β_j is achieved.

B. Information on λ_i^* and Incomplete ϕ_i^* Available

It is assumed that the information about incomplete DOF readings of the ith mode shape for the modified structural dynamic system $\psi_i^{a^*}$ is available, that is, only NAI($<N$) DOF readings of the total N DOF readings exist for the ith measured mode. The measured incomplete mode shape for the modified system $\psi_i^{a^*}$ should be paired to the mode shape for the original system ϕ_i by using MAC factors, as defined in Eq. (43).

The ith eigenvector for the modified system then can be expressed by

$$\phi_i^* = \phi_i^{a^*} + \phi_i^{u^*} \tag{52}$$

where the scaled vector $\phi_i^{a^*}$ is defined as

$$\phi_i^{a^*} = \beta_i \psi_i^{a^*} \tag{53}$$

in which the mode scale factor (MSF) υ_i is calculated from

$$\upsilon_i = \frac{\phi_i^{a^T}\psi_i^{a^*}}{\psi_i^{a^{*T}}\psi_i^{a^*}} \tag{54}$$

The remaining components for the modified mode $\phi_i^{u^*}$ are computed from

$$\phi_i^{u^*} = \phi_i^u + \Delta\phi_i^u \tag{55}$$

where the change of unknown DOF readings, $\Delta\phi_i^u$, after rewriting Eq. (22), is

$$\Delta\phi_i^u = \sum_{l=1,l\neq i}^{\mathrm{NC}} C_{il}\phi_l^u \tag{56}$$

Consequently, the ith complete eigenvector for the modified system ϕ_i^* is obtained from

$$\phi_i^* = \varphi_i^a + \sum_{l=1,l\neq i}^{\mathrm{NC}} C_{il}\phi_i^u \tag{57}$$

where φ_i^a, by utilizing Eqs. (53) and (55), is defined as

$$\varphi_i^a = \upsilon_i\psi_i^{a^*} + \phi_i^u \tag{58}$$

Note that the MSF υ_i in Eq. (58) has to be updated for each iteration (if an iterative procedure is required) because $\phi_i^{a^*}$ must be scaled in such a way as to be close to $\phi_i^{u^*}$.

Rewriting Eq. (21), by using Eq. (16), leads to

$$\sum_{k=1}^{N} \frac{\phi_k^T\left[\Delta K - \lambda_i^*\Delta M\right]\phi_i^*}{\left(\lambda_i^* - \lambda_k\right)}\phi_k^a - \phi_i^{a^*} = 0 \tag{59}$$

Note that Eq. (59) comprises a total of NA equations. The formulation is also suitable for special cases that all DOF readings for the modified system are available, that is, the modified mode shape is complete.

By the use of Eqs. (44) and (45), Eq. (59) can be expressed as

$$\sum_{j=1}^{\mathrm{NP}} a_{ji}^{K^*}\alpha_j - \lambda_i^*\sum_{j=1}^{\mathrm{NP}} a_{ji}^{M^*}\beta_j - \phi_i^{a^*} = 0 \tag{60}$$

where the sensitivity coefficient vectors are defined as

$$\boldsymbol{a}_{ji}^{K^*} = \sum_{k=1}^{N}\frac{\phi_k^T K_j\phi_i^*}{\left(\lambda_i^* - \lambda_k\right)}\phi_k^a, \qquad \boldsymbol{a}_{ji}^{M^*} = \sum_{k=1}^{N}\frac{\phi_k^T M_j\phi_i^*}{\left(\lambda_i^* - \lambda_k\right)}\phi_k^a \tag{61}$$

in which the modified eigenvectors ϕ_i^* are calculated using Eq. (57) and the mode participation factors C_{ik} can be obtained from Eq. (50).

The set of nonlinear equations (60) and (50) forms a basis for an iterative solution procedure to solve for system parameters α_j and β_j. Similarly, the initial mode participation factors C_{ik} are assumed to be equal to zero. A first estimate for α_j and β_j is then obtained from Eq. (60). The filtered SVD technique is often required. The next estimate for the mode participation factors C_{ik} is obtained from

Eq. (50). The Jacobi transformation method might be required to improve convergence performance. The modified eigenvectors ϕ_i^* then can be calculated using Eq. (57). Therefore, Eqs. (50) and (60), as well as Eq. (57), are used recursively to compute further estimate for α_j and β_j as well as C_{ik} until the condition of convergence for α_j and β_j is satisfied.

V. Numerical Examples

A. Plane Frame Model Problem

A symmetric plane frame shown in Fig. 1 is used to demonstrate the effectiveness of the proposed techniques for eigendata modification and model updating. The results obtained from the proposed techniques are also used to compare those obtained from first-order approximation method. To avoid problems associated with structural symmetry, a nonsymmetric element mesh with 18 elements, 18 nodes with a total of 48 DOF, is generated. All structural members have the same material and geometric properties with Young's modulus $E = 2.1 \times 10^{11}$ N/m^2, density $\rho = 7800$ kg/m^3, cross-sectional area $A = 0.092$ m^2, and second moment of area $I = 4.52 \times 10^{-5}$ m^4. The geometry of the structure and element numbering are shown in Fig. 1. A hypothetical sensor set scenario, placed at nodes 3, 5, 7, 9, and 11–14 and measuring only translation displacement readings in the indicated directions, is also shown in Fig. 1.

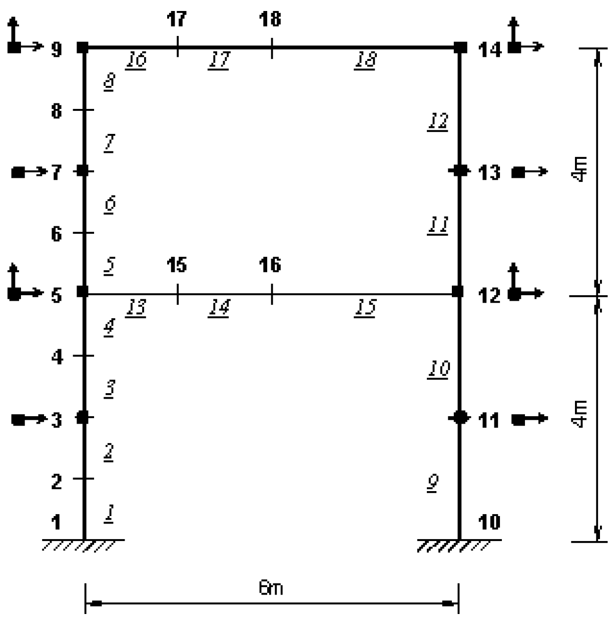

Fig. 1 Symmetric model plane frame problem.

The convergence performance of the proposed techniques for eigendata modification is demonstrated in Table 1. In this case, the stiffness values of elements 3, 11, and 16 were modified by factors of +80, −50, and +30%, respectively, and two 400- and two 200-kg masses were attached to nodes 5, 12, 9, and 14, respectively. Therefore, changes in both stiffness and mass matrices were required to correct the model. The modal properties of the modified structure were computed using the procedure for eigendata modification described earlier. The natural frequency estimates of the first 15 modes through a succession of iterations are listed in Table 1 along with the exact solution. The results given at the first iteration are identical to those from first-order approximation method. The parings of the original modes and the modified modes were assured by using MAC values, as defined in Eq. (43). The MAC diagonal values given in Table 1 show that the modified modes match very well the corresponding original modes.

The results in Table 1 show that the proposed technique achieves convergence after only a few iterations. The first-order approximation method may not be sufficient to predict the modal parameters even if the modification of structural parameters is not very large.

The effectiveness of the proposed techniques for eigendata modification with respect to the number of original eigenvectors is investigated in the case with relatively small perturbations of structural parameters, as shown in Table 2, where the case for both stiffness and masses at elements 3, 11, and 16 modified by factors +30, −20, and +10% is considered. It is found that only a limited knowledge of the original eigenvectors is required; even a total number of 24 original eigenvectors (half the number of all original eigenvectors) is sufficient to predict well the modifications of natural frequencies for the case with relatively small perturbations of structural parameters. It is also found that excellent results can be obtained when all 48 original modes are adopted in calculations.

The effect of the number of original modes adopted for the case with relatively large perturbations of structural parameters is also investigated, as shown in Table 3. Similar conclusions can be drawn from the given results, although a slightly increasing number of original modes may be required to ensure that good predictions are obtained in the case considered.

The results in Tables 4 and 5 demonstrate the effectiveness and the convergence performance of the proposed techniques for model updating. It is assumed that stiffness values at elements 3, 11, and 16 are corrupted by factors +50, −30, and +20%, respectively. Table 4 shows the adjusted natural frequencies at different iteration numbers, where information about only eight correct natural frequencies is employed to adjust the corrupted model. From the results, it can be seen that the convergence of the proposed techniques for model updating using information about only correct natural frequencies is achieved rapidly. The first eight adjusted natural frequencies are in excellent agreement with the exact solutions, whereas the other

Table 1 Predicted natural frequencies (hertz) at different iteration numbers for eigendata modification

			First iteration		Second iteration		Sixth iteration		
Original	Modified (exact)	Exact $\Delta\omega$	Predicted ω	Predicted $\Delta\omega$	Predicted ω	Predicted $\Delta\omega$	Predicted ω	Predicted $\Delta\omega$	MAC diagonal value
1.4074	1.3652	−0.0422	1.3702	−0.0372	1.3654	−0.0420	1.3652	−0.0422	0.9999
4.8024	4.4557	−0.3467	4.5307	−0.2717	4.4555	−0.3469	4.4557	−0.3467	0.9969
7.2706	7.2297	−0.0409	7.2934	0.0228	7.2289	−0.0417	7.2297	−0.0409	0.9971
9.0599	8.9326	−0.1273	9.0003	−0.0596	8.9325	−0.1274	8.9326	−0.1273	0.9937
16.6852	16.0454	−0.6398	16.7982	0.1130	16.0423	−0.6429	16.0454	−0.6398	0.8451
19.1003	18.9085	−0.1918	19.0853	−0.0150	18.9053	−0.1950	18.9085	−0.1918	0.8565
24.1619	25.0955	0.9336	25.4847	1.3228	25.0940	0.9321	25.0955	0.9336	0.9820
24.2985	23.1432	−1.1553	23.5635	−0.7350	23.1460	−1.1525	23.1432	−1.1553	0.9127
30.9663	30.9524	−0.0139	31.1554	0.1891	30.9474	−0.0189	30.9524	−0.0139	0.9198
32.8360	32.1593	−0.6767	32.3480	−0.4880	32.1595	−0.6765	32.1593	−0.6767	0.9270
51.2520	50.4625	−0.7895	51.0786	−0.1734	50.4549	−0.7971	50.4625	−0.7895	0.9551
61.0721	61.0091	−0.0630	61.9479	0.8758	61.0002	−0.0719	61.0091	−0.0630	0.9348
64.8405	63.5479	−1.2926	64.6490	−0.1915	63.5425	−1.2980	63.5479	−1.2926	0.9011
72.6380	75.2733	2.6353	76.0621	3.4241	75.2754	2.6374	75.2733	2.6353	0.8629
85.3921	83.1801	−2.2120	84.6220	−0.7701	83.1423	−2.2498	83.1801	−2.2120	0.7478
$D\lambda$[a]				$3.43E{-2}$		$2.81E{-2}$		$1.67E{-7}$	

[a] $D\lambda = \sum_{i=1}^{\mathrm{NL}} (|d\lambda_i|/|\lambda_i|)$.

Table 2 Predicted natural frequencies (hertz) obtained from various number of original eigenvectors adopted: case with relatively small perturbations

			16 Modes		24 Modes		36 Modes		48 Modes (all)	
Original	Modified (exact)	Exact $\Delta\omega$	Predicted ω	Predicted $\Delta\omega$	Predicted ω	Predicted ω	Predicted ω	Predicted $\Delta\omega$	Predicted ω	Predicted $\Delta\omega$
1.4074	1.4077	0.0003	1.4080	0.0006	1.4078	0.0004	1.4077	0.0003	1.4077	0.0003
4.8024	4.7544	−0.0480	4.7568	−0.0456	4.7554	−0.0470	4.7544	−0.0480	4.7544	−0.0480
7.2706	7.2615	−0.0091	7.2644	−0.0062	7.2630	−0.0076	7.2615	−0.0091	7.2615	−0.0091
9.0599	9.0127	−0.0472	9.0155	−0.0444	9.0141	−0.0458	9.0128	−0.0471	9.0127	−0.0472
16.6852	16.6709	−0.0143	16.6861	0.0009	16.6784	−0.0068	16.6709	−0.0143	16.6709	−0.0143
19.1003	19.2176	0.1173	19.2408	0.1405	19.2261	0.1258	19.2177	0.1174	19.2176	0.1173
24.1619	23.8204	−0.3415	23.8577	−0.3042	23.8414	−0.3205	23.8209	−0.3410	23.8204	−0.3415
24.2985	23.9807	−0.3178	23.9975	−0.3010	23.9874	−0.3111	23.9811	−0.3174	23.9808	−0.3177
30.9663	30.8394	−0.1269	30.8594	−0.1069	30.8523	−0.1140	30.8403	−0.1260	30.8394	−0.1269
32.8360	32.6917	−0.1443	32.7168	−0.1192	32.6967	−0.1393	32.6918	−0.1442	32.6917	−0.1443
51.2520	51.4690	0.2170	51.4852	0.2332	51.4813	0.2293	51.4704	0.2184	51.4690	0.2170
61.0721	61.1884	0.1163	61.2543	0.1822	61.2395	0.1674	61.1913	0.1192	61.1884	0.1163
64.8405	64.3895	−0.4510	64.4238	−0.4167	64.4003	−0.4402	64.3902	−0.4503	64.3895	−0.4510
72.6380	72.3282	−0.3098	72.4333	−0.2047	72.4142	−0.2238	72.3311	−0.3069	72.3282	−0.3098
85.3921	85.4386	0.0465	85.5038	0.1117	85.4669	0.0748	85.4520	0.0599	85.4386	0.0465

Table 3 Predicted natural frequencies (hertz) obtained from various number of original eigenvectors adopted: case with relatively large perturbations

			16 Modes		24 Modes		36 Modes		48 Modes (all)	
Original	Modified (exact)	Exact $\Delta\omega$	Predicted ω	Predicted $\Delta\omega$	Predicted ω	Predicted ω	Predicted ω	Predicted $\Delta\omega$	Predicted ω	Predicted $\Delta\omega$
1.4074	1.3652	−0.0422	1.3676	−0.0398	1.3657	−0.0417	1.3652	−0.0422	1.3652	−0.0422
4.8024	4.4557	−0.3467	4.4721	−0.3303	4.4603	−0.3421	4.4558	−0.3466	4.4557	−0.3467
7.2706	7.2297	−0.0409	7.2567	−0.0139	7.2386	−0.0320	7.2301	−0.0405	7.2297	−0.0409
9.0599	8.9326	−0.1273	8.9502	−0.1097	8.9388	−0.1211	8.9331	−0.1268	8.9326	−0.1273
16.6852	16.0454	−0.6398	16.2543	−0.4309	16.0649	−0.6203	16.0461	−0.6391	16.0454	−0.6398
19.1003	18.9085	−0.1918	19.1049	0.0046	18.9679	−0.1324	18.9092	−0.1911	18.9085	−0.1918
24.1619	25.0955	0.9336	25.3535	1.1916	25.2468	1.0849	25.0976	0.9357	25.0955	0.9336
24.2985	23.1432	−1.1553	23.3164	−0.9821	23.1614	−1.1371	23.1454	−1.1531	23.1432	−1.1553
30.9663	30.9524	−0.0139	31.1866	0.2203	31.0692	0.1029	30.9563	−0.0100	30.9524	−0.0139
32.8360	32.1593	−0.6767	32.3539	−0.4821	32.1922	−0.6438	32.1604	−0.6756	32.1593	−0.6767
51.2520	50.4625	−0.7895	50.6016	−0.6504	50.5141	−0.7379	50.4692	−0.7828	50.4625	−0.7895
61.0721	61.0091	−0.0630	61.4802	0.4081	61.2478	0.1757	61.0167	−0.0554	61.0091	−0.0630
64.8405	63.5479	−1.2926	64.0560	−0.7845	63.6111	−1.2294	63.5522	−1.2883	63.5478	−1.2927
72.6380	75.2733	2.6353	76.5691	3.9311	76.2504	3.6124	75.3005	2.6625	75.2733	2.6353
85.3921	83.1801	−2.2120	84.5678	−0.8243	83.4236	−1.9685	83.2664	−2.1257	83.1801	−2.2120

natural frequency estimates are closer to the exact solutions compared to the results from the first approximation method.

In Table 5, the results show the adjusted natural frequencies obtained using information about three incomplete correct modes (modes 1, 2, and 3) with DOF readings measured at the set of sensor scenario, as shown in Fig. 1. Here, the stiffness values at elements 3, 11, and 16 are assumed to be corrupted by factors +50, −30, and +20%, respectively. Again, the adjusted natural frequencies obtained using the proposed techniques are very accurate when compared the exact solution. It can be seen that the first-order approximation method, the results shown in Table 5 under the heading of first iteration, may not be sufficient for model updating in this case.

Furthermore, both stiffness and mass can be adjusted at the same time using the proposed techniques, as summarized in Table 6. It is assumed that the exact stiffness and mass for each element have factors of unity, respectively, and both stiffness and mass values at elements 3, 5, 8, 9, 11, and 16 are corrupted by factors +30, −10, +10, −10, −20, and +10%, respectively. Various number of incomplete correct modes with DOF readings measured at the assumed sensor set are employed to adjust the corrupted model. It can be seen that both stiffness and mass can be adjusted correctly by using information about only six incomplete correct modes.

B. Kabe's Problem

An eight-DOF mass–spring system introduced by Kabe,[8] as shown in Fig. 2, is adopted to investigate the effectiveness of the proposed techniques for large modifications of structural parameters. The structure represents a severe test case because there are large relative differences in the magnitudes of some of the stiffness matrix coefficients and all eight natural frequencies are very close to each other.

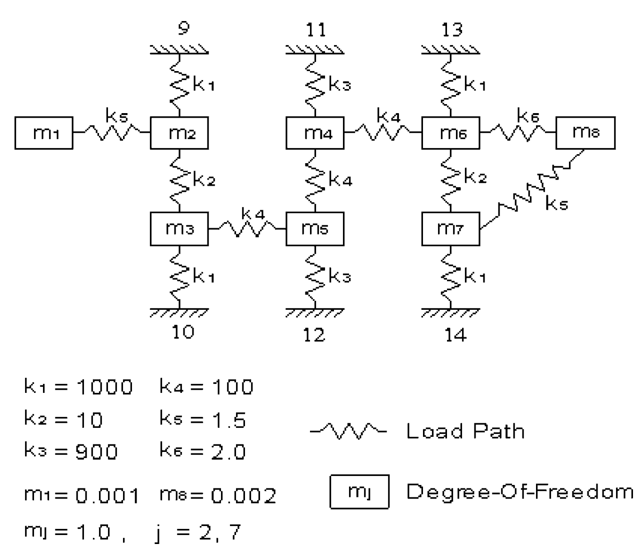

Fig. 2 Kabe's problem.

The first case considered is the effect of large modifications of structural parameters on the proposed techniques for eigendata modification, as shown in Tables 7 and 8. The results shown in Table 7 are the first three modified natural frequencies and the corresponding mode shapes, where the modifications of stiffness values $\Delta k_1 = +500$, $\Delta k_2 = 0$, $\Delta k_3 = -450$, $\Delta k_4 = +100$, $\Delta k_5 = +0.5$, and

Table 4 Adjusted natural frequencies (hertz) at different iteration numbers for model updating by using only eight correct frequencies

	Adjusted natural frequencies				
Corrupted	First iteration	Second iteration	Fifth iteration	Exact	MAC value
1.4096	1.4062	1.4074	1.4074	1.4074[a]	1.0000
4.7367	4.7958	4.8031	4.8024	4.8024[a]	0.9990
7.2662	7.2607	7.2709	7.2706	7.2706[a]	0.9992
9.0017	9.0515	9.0605	9.0599	9.0599[a]	0.9975
16.5695	16.6250	16.6849	16.6851	16.6852[a]	0.9485
19.0584	19.0589	19.1079	19.1003	19.1003[a]	0.9475
23.7686	24.2738	24.3031	24.2984	24.2985[a]	0.9917
24.8744	24.0807	24.1633	24.1620	24.1619[a]	0.9775
31.1452	31.0702	31.0542	31.0471	30.9663	0.9904
32.5375	32.7893	32.8204	32.8064	32.8360	0.9878
50.9666	51.3042	51.3628	51.3394	51.2520	0.9843
61.3211	60.6836	60.8501	60.8391	61.0721	0.9666
64.4138	65.0944	65.0988	65.0533	64.8405	0.9498
74.7171	72.7126	72.9228	72.9089	72.6380	0.9483
84.5668	85.1241	85.2344	85.1871	85.3921	0.9063
$D\alpha\beta$[b]	1.23E+0	2.50E−01	1.79E−03	/	/

[a]Values of exact frequencies were used for model updating.
[b]$D\alpha\beta = \sum_{j=1}^{NP} |\Delta\alpha_j| + |\Delta\beta_j|$.

Table 5 Adjusted natural frequencies (hertz) at different iteration numbers for model updating by using three incomplete correct modes

	Adjusted natural frequencies				
Corrupted	First iteration	Second iteration	Fifth iteration	Exact	MAC value
1.4096	1.4055	1.4071	1.4074	1.4074[a]	1.0000
4.7367	4.7737	4.8020	4.8024	4.8024[a]	0.9990
7.2662	7.2409	7.2724	7.2706	7.2706[a]	0.9992
9.0017	9.0142	9.0647	9.0599	9.0599	0.9975
16.5695	16.5499	16.7183	16.6852	16.6852	0.9485
19.0584	18.9513	19.1244	19.1004	19.1003	0.9475
23.7686	24.2469	24.3260	24.2986	24.2985	0.9917
24.8744	23.9801	24.2612	24.1617	24.1619	0.9775
31.1452	30.7421	31.0416	30.9663	30.9663	0.9904
32.5375	32.6977	32.8205	32.8359	32.8360	0.9878
50.9666	50.9483	51.3226	51.2521	51.2520	0.9843
61.3211	60.7231	61.3023	61.0724	61.0721	0.9666
64.4138	64.3805	64.8695	64.8402	64.8405	0.9498
74.7171	71.6284	72.9868	72.6377	72.6380	0.9483
84.5668	84.7122	85.4616	85.3914	85.3921	0.9063
$D\alpha\beta$[b]	1.65E+00	8.08E−01	4.40E−03	/	/

[a]Incomplete modes were used for model updating. [b]$D\alpha\beta = \sum_{j=1}^{NP} |\Delta\alpha_j| + |\Delta\beta_j|$.

$\Delta k_6 = +2.0$ are considered. Some stiffness values were increased by a maximum of 100%, whereas others were kept the same. From the results, it can be seen that the first three modified natural frequencies are determined exactly. Each predicted DOF reading of all three modes is in excellent agreement with the exact solution; even the changes of most DOF readings are very large. Therefore, the modes of the modified system can be exactly calculated using the developed theory and the computational procedure. Again, the MAC values are employed to check the pairings of the original modes and the modified modes.

The effect of large modifications in both stiffness and mass is also investigated, as summarized in Table 8. In this case, the modifications of stiffness are the same as those discussed earlier, and masses are modified with values of $\Delta m_2 = +1.0$, $\Delta m_4 = -0.5$, $\Delta m_6 = -0.5$, and $\Delta m_8 = -0.001$. Both the modifications of stiffness and mass are significant compared to the original values. The changes of natural frequencies are large, and some of natural frequencies are closer to neighboring frequency values than to the modified values. It is obvious that the first-order approximation method is insufficient to predict the modified natural frequencies. Again, the modified natural frequencies caused by large modifications of both stiffness and mass are exactly determined using the proposed techniques for eigendata modification.

Kabe's problem is now utilized to investigate the effectiveness of the proposed techniques for model updating. It is assumed that the load path stiffness is corrupted at different levels, as listed in Table 9 under the heading of corrupted stiffness value. These corrupted stiffness values were obtained by modifying the exact stiffness values with $\Delta k_1 = +500$, $\Delta k_2 = 0$, $\Delta k_3 = -450$, $\Delta k_4 = +100$, $\Delta k_5 = +0.5$, and $\Delta k_6 = +2.0$. There are 14 independent terms to be adjusted with the preservation of connectivity.

The results obtained using one complete mode (mode 1) show that many of the adjusted stiffness coefficients have significantly approached the exact values. Furthermore, information about two correct modes (complete and/or incomplete) is considered to adjust the corrupted analytical model. The adjusted stiffness coefficients become much closer to the exact values when information about complete mode 1 and incomplete mode 2 missing DOF readings at nodes 1, 5, and 7 is utilized. Excellent adjusted stiffness coefficients are obtained when information about complete mode 1 and incomplete mode 3 missing DOF readings at nodes 1 and 8 is adopted. Finally, the stiffness values are exactly determined using information about two complete modes, that is, mode 2 and mode 3.

The natural frequencies for cases with the same stiffness modifications and various modal data utilized for model updating as discussed earlier are compared with the exact values, as listed in Table 10. It can be seen that the natural frequency of complete mode 1, which was adopted for model updating, is reproduced

Table 6 Both stiffness and mass factors adjusted at the same time with various numbers of incomplete correct modes

	Adjusted structural parameters									
	Corrupted		2 Modes		4 Modes		6 Modes		Exact	
Element	Stiffness	Mass	Stiffness	Mass	Stiffness	Mass	Stiffness	Mass	Stiffness	Mass
1	1.00	1.00	0.98	1.00	0.99	0.99	0.99	1.00	1.00	1.00
2	1.00	1.00	0.89	0.99	0.97	0.92	0.99	0.99	1.00	1.00
3	1.30	1.30	1.14	1.26	1.01	1.06	0.99	0.99	1.00	1.00
4	1.00	1.00	1.01	0.97	1.00	0.96	0.99	0.99	1.00	1.00
5	0.90	0.90	0.99	0.89	0.99	0.88	0.99	0.99	1.00	1.00
6	1.00	1.00	0.94	1.03	0.98	1.02	0.99	0.99	1.00	1.00
7	1.00	1.00	1.01	1.02	1.00	1.03	0.99	0.99	1.00	1.00
8	1.10	1.10	0.97	1.08	0.99	0.94	0.99	0.99	1.00	1.00
9	0.90	0.90	0.99	0.91	0.99	0.93	0.99	0.99	1.00	1.00
10	1.00	1.00	0.99	1.06	1.00	1.03	0.99	0.99	1.00	1.00
11	0.80	0.80	0.96	0.89	0.99	0.98	0.99	0.99	1.00	1.00
12	1.00	1.00	0.98	1.04	1.00	1.01	0.99	0.99	1.00	1.00
13	1.00	1.00	1.04	0.95	1.00	1.03	0.99	0.99	1.00	1.00
14	1.00	1.00	1.02	0.97	1.01	0.98	0.99	0.99	1.00	1.00
15	1.00	1.00	0.95	1.02	0.99	1.01	0.99	0.99	1.00	1.00
16	1.10	1.10	1.04	0.96	1.00	0.99	0.99	0.99	1.00	1.00
17	1.00	1.00	1.06	0.93	1.01	1.00	0.99	0.99	1.00	1.00
18	1.00	1.00	0.94	0.98	0.99	1.00	0.99	0.99	1.00	1.00

Table 7 Predicted first three natural frequencies (hertz) and corresponding mode shapes obtained from eigendata modification techniques when only stiffness is modified

	Mode 1			Mode 2			Mode 3		
	Original $\omega = 4.8805$	Modified (exact) = 3.9421	Predicted $\omega = 3.9421$	Original $\omega = 5.0469$	Modified (exact) = 6.1713	Predicted $\omega = 6.1713$	Original $\omega = 5.0552$	Modified (exact) = 6.1846	Predicted $\omega = 6.1846$
DOF 1	0.1436	0.0020	0.0020	2.7764	3.9935	3.9935	1.0967	−0.0225	−0.0226
DOF 2	0.0536	0.0014	0.0014	0.9152	0.9912	0.9912	0.3591	−0.0055	−0.0055
DOF 3	0.3595	0.1269	0.1269	0.1266	0.0364	0.0364	−0.0710	0.0034	0.0034
DOF 4	0.6056	0.6956	0.6956	−0.0882	0.0027	0.0028	−0.0271	−0.0157	−0.0157
DOF 5	0.6046	0.6956	0.6956	0.0407	−0.0120	−0.0120	−0.1077	0.0037	0.0037
DOF 6	0.3623	0.1268	0.1268	−0.1240	0.0030	0.0030	0.0830	0.0480	0.0480
DOF 7	0.0616	0.0016	0.0017	−0.3368	0.0054	0.0054	0.9184	0.9982	0.9982
DOF 8	0.5045	0.1070	0.1070	−0.5059	0.0076	0.0076	1.0414	0.7343	0.7343
MAC value		0.7570			0.9484			0.5444	

Table 8 Predicted natural frequencies (hertz) obtained from eigendata modification techniques when both stiffness and mass are modified

				First-order approximation			Proposed technique			MAC
Original	Modified (exact)	Exact $\Delta\omega$	Change in %	Predicted ω	Predicted $\Delta\omega$	Difference in %	Predicted ω	Predicted $\Delta\omega$	Difference in %	diagonal value
4.8805	4.2600	−0.6205	−12.71	4.6535	−0.2270	63.42	4.2600	−0.6205	0.00	0.7023
5.0469	4.3714	−0.6755	−13.38	4.5386	−0.5083	24.75	4.3714	−0.6755	0.00	0.9017
5.0553	6.1872	1.1319	22.39	5.7955	0.7402	34.61	6.1872	1.1319	0.00	0.4795
5.1505	5.2315	0.0810	1.57	5.5271	0.3766	364.94	5.2315	0.0810	0.00	0.6354
5.4420	6.6170	1.1750	21.59	6.0395	0.5975	49.15	6.6170	1.1750	0.00	0.4078
5.6595	6.8914	1.2319	21.77	6.3503	0.6908	43.92	6.8914	1.2319	0.00	0.5891
6.1734	7.1205	0.9471	15.34	7.0980	0.9246	2.38	7.1205	0.9471	0.00	1.0000
6.6668	12.3313	5.6645	84.97	12.2704	5.6036	1.08	12.3313	5.6645	0.00	1.0000

Table 9 Adjusted stiffness and mass values obtained from model updating techniques by using information about various modal data used

		Adjusted stiffness value				
Load path location	Corrupted stiffness value	Complete mode 1	Modes 1 and incomplete 2	Modes 1 and incomplete 3	Complete modes 2 and 3	Exact stiffness value
$k_{1\text{-}2}$ (k_5)	2.0	1.5	1.5	1.5	1.5	1.5
$k_{2\text{-}9}$ (k_1)	1500.0	1000.7	1000.1	1000.1	1000.0	1000.0
$k_{2\text{-}3}$ (k_2)	10.0	10.1	10.0	10.0	10.0	10.0
$k_{3\text{-}10}$ (k_1)	1500.0	1092.9	1000.2	999.6	1000.0	1000.0
$k_{3\text{-}5}$ (k_4)	200.0	236.5	100.3	99.5	100.0	100.0
$k_{4\text{-}11}$ (k_3)	450.0	845.1	899.8	900.0	900.0	900.0
$k_{4\text{-}5}$ (k_4)	200.0	200.0	100.0	100.7	100.0	100.0
$k_{5\text{-}12}$ (k_3)	450.0	844.8	899.9	900.2	900.0	900.0
$k_{4\text{-}6}$ (k_4)	200.0	236.2	100.5	100.0	100.0	100.0
$k_{6\text{-}13}$ (k_1)	1500.0	1091.9	1000.7	1000.1	1000.0	1000.0
$k_{6\text{-}7}$ (k_2)	10.0	10.1	9.6	10.0	10.0	10.0
$k_{7\text{-}14}$ (k_1)	1500.0	998.3	998.3	1000.1	1000.0	1000.0
$k_{6\text{-}8}$ (k_6)	4.0	3.0	1.9	2.0	2.0	2.0
$k_{7\text{-}8}$ (k_5)	2.0	1.2	1.5	1.5	1.5	1.5

Table 10 Adjusted natural frequencies obtained from model updating techniques by using information about various modal data

	Adjusted natural frequency, Hz				
Corrupted	Complete mode 1	Modes 1 and incomplete 2	Modes 1 and incomplete 3	Complete modes 2 and 3	Exact
3.9422	4.8805[a]	4.8805[a]	4.8805[a]	4.8805	4.8805
5.0181	5.0516	5.0458[a]	5.0469	5.0469[a]	5.0469
6.1714	5.0551	5.0511	5.0553[a]	5.0553[a]	5.0553
6.1847	5.4279	5.1505	5.1512	5.1505	5.1505
6.6500	6.1184	5.4440	5.4407	5.4420	5.4420
6.6869	6.1735	5.6611	5.6607	5.6595	5.6595
7.1320	6.4762	6.1734	6.1734	6.1734	6.1734
8.7283	7.2619	6.5910	6.6668	6.6667	6.6667

[a]Modes corresponding to frequency values used for model updating.

exactly by the adjusted mode. The other natural frequencies become closer to the exact values when more information about modal data is utilized. As expected, the exact natural frequencies are obtained if information about two complete modes (modes 2 and 3) is employed to adjust the corrupted model.

VI. Conclusions

A general nonlinear perturbation theory is developed that can be used for eigendata modification, model updating, and damage identification. The computational procedures for forward problems, such as eigendata modification, and for inverse problems, such as model updating and damage identification, are presented based on the developed general theory. The Jacobi transformation method and the accelerated modal method are introduced to improve the convergence performance of the proposed iterative procedures. Depending on the information about modal data available, such as only modified natural frequencies and incomplete modified modal data, different computational techniques are proposed for model updating and damage identification, without requirement of model reduction or mode shape expansion. The approximation caused by model reduction or mode shape expansion then can be avoided.

Two numerical examples, a plane frame model problem and Kabe's problem, are employed to demonstrate the effectiveness of the proposed techniques. It has been shown that the convergence of the proposed techniques can be achieved rapidly. The exact modal parameters of a modified structural dynamic system can be obtained using the proposed techniques for eigendata modification. The corrupted model can be adjusted correctly, and the adjusted model reproduces exactly the modal data adopted in the calculation using the proposed techniques for model updating. The results obtained, therefore, show that the developed theory and the computational procedure can provide exact relationship between the perturbation of structural parameters and the perturbation of modal parameters. Only a limited knowledge of original modes is required to produce correctly the modifications of modal parameters and the modifications of structural parameters in both cases with relatively small and relatively large perturbations present, which is very useful for dynamic systems with large number of DOF. It is found that the first-order approximation method may not be sufficient for eigendata modification and for model updating when relatively large modifications of structural parameters are present in the system considered.

References

[1]Fox, R. L., and Kapoor, M. P., "Rates of Changes of Eigenvalues and Eigenvectors," *AIAA Journal*, Vol. 6, No. 12, 1968, pp. 2426–2429.

[2]Rudisill, C. S., and Bhatia, K. G., "Second Derivatives of the Flutter Velocity and the Optimization of Aircraft Structures," *AIAA Journal*, Vol. 10, No. 12, 1972, pp. 1569–1572.

[3]Wang, J., Heylen, W., and Sas, P., "Accuracy of Structural Modification Techniques," *Proceedings of the 5th International Modal Analysis Conference*, Union College, Schenectady, NY, 1987, pp. 65–75.

[4]To, W. M., and Ewins, D. J., "Non-Linear Sensitivity Analysis of Mechanical Structures Using Modal Data," *Journal of Mechanical Engineering Science*, Vol. 205, No. C1, 1991, pp. 67–75.

[5]Beliveau, J. G., Cogan, S., Lallement, G., and Ayer, F., "Iterative Least-Squares Calculation for Modal Eigenvector Sensitivity," *AIAA Journal*, Vol. 34, No. 2, 1996, pp. 385–391.

[6]Berman, A., and Nagy, E. J., "Improvement of a Large Analytical Model Using Test Data," *AIAA Journal*, Vol. 21, No. 8, 1983, pp. 1168–1173.

[7]Caesar, B., and Peter, J., "Direct Update of Dynamic Mathematical Models from Modal Test Data," *AIAA Journal*, Vol. 25, No. 11, 1987, pp. 1494–1499.

[8]Kabe, A. M., "Stiffness Matrix Adjustment Using Modal Data," *AIAA Journal*, Vol. 23, No. 9, 1985, pp. 1431–1436.

[9]Kammer, D. C., "Optimum Approximation for Residual Stiffness in Linear System Identification," *AIAA Journal*, Vol. 26, No. 1, 1988, pp. 104–112.

[10]Smith, S. W., and Beattie, C. A., "Secant-Method Adjustment for Structural Models," *AIAA Journal*, Vol. 29, No. 1, 1991, pp. 119–126.

[11]Ladeveze, P., Nedjar, D., and Reynier, M., "Updating of Finite Element Models Using Vibration Tests," *AIAA Journal*, Vol. 32, No. 7, 1994, pp. 1485–1491.

[12]Bicanic, N., and Chen, H. P., "Damage Identification in Framed Structures Using Natural Frequencies," *International Journal for Numerical Methods in Engineering*, Vol. 40, No. 23, 1997, pp. 4451–4468.

[13]Chen, H. P., and Bicanic, N., "Assessment of Damage in Continuum Structures Based on Incomplete Modal Information," *Computers and Structures*, Vol. 74, No. 5, 2000, pp. 559–570.

[14]Link, M., "Identification and Correction of Errors in Analytical Models Using Test Data—Theoretical and Practical Bounds," *Proceedings of the 8th International Modal Analysis Conference*, Scholarly Communications, Virginia Polytechnic Inst. and State Univ., Blacksburg, VA, 1990, pp. 570–578.

[15]Cawley, P., and Adams, R. D., "The Location of Defects in Structures from Measurements of Natural Frequencies," *Journal of Strain Analysis*, Vol. 14, No. 2, 1979, pp. 49–57.

[16]Ojalvo, I. U., and Zhang, L. M., "Tutorial: Fundamentals of Ill-Conditioning," *International Journal of Analytical and Experimental Modal Analysis*, Vol. 8, No. 2, 1993, pp. 165–176.

A. Berman
Associate Editor